The Calculation Of Declining And Inclining Sundials - An Unusual Approach

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The mathematical tools of vector analysis, differential calculus and trigonometry are here used to deduce the formulae concerning planar dials in any spatial position. The intention was to get clear equations for the calculation of the shadow coordinates, *i.e* they should contain only the astronomical coordinates of the sun's azimuth and altitude or sun's declination and hour angle as well as the dial's design parameters geographical latitude, pin gnomon length, declination and inclination. A diagram represents graphically the course of the calculated declination and hour lines of four typical dial types.

Fundamentals for calculation

Horizon system ⇔ equator system

Symbols used: a = azimuth of the sun ($a = 0°$ = at the local meridian, positive values west of the meridian), h = altitude of the sun, δ = sun's declination, τ = hour angle ($\tau = 0°$ = midday, positive values west of the meridian), ϕ = geographical latitude.

(1) $\sin a \cos h = \cos\delta \sin\tau$,

(2) $\cos a \cos h = \sin\phi \cos\delta \cos\tau - \cos\phi \sin\delta$,

(3) $\tan a = \dfrac{\cos\delta \sin\tau}{\sin\phi \cos\delta \cos\tau - \cos\phi \sin\delta}$

(4) $\sin h = \sin\phi \sin\delta + \cos\phi \cos\delta \cos\tau$,

Note: When using equation (3) it is important to notice if the arctangent function changes its sign: choose the relevant quadrant.

Sun's vector

Using the well known correlations between the spherical coordinates r (length of the radius), ϑ (angle that r makes with the z' axis) and η (angle that the projection of r onto the $x'y'$ plane makes with the x' axis) of a space point and the Cartesian coordinates (right-handed system) $x' = r \sin\vartheta \cos\eta$, $y' = r \sin\vartheta \sin\eta$ and $z' = r \cos\vartheta$ with the assignments $r = S$ for the distance coordinate and $\vartheta = 90° - h$, $\eta = 270° - a$ for the angle coordinates, the sun's vector can be written in the column notation

(5) $$\vec{S} = \begin{pmatrix} x'_S \\ y'_S \\ z'_S \end{pmatrix} = S \begin{pmatrix} -\cos h \sin a \\ -\cos h \cos a \\ \sin h \end{pmatrix}.$$

Transformation of coordinates of the sun's vector by rotating about the applicate and abscissa

Rotating the sun's vector $\vec{S}$ according to (5) through the angle d about the z' axis yields the vector

(6) $$\begin{pmatrix} \cos d & -\sin d & 0 \\ \sin d & \cos d & 0 \\ 0 & 0 & 1 \end{pmatrix} S \begin{pmatrix} -\cos h \sin a \\ -\cos h \cos a \\ \sin h \end{pmatrix} = S \begin{pmatrix} -\cos d \cos h \sin a + \sin d \cos h \cos a \\ -\sin d \cos h \sin a - \cos d \cos h \cos a \\ \sin h \end{pmatrix} = \begin{pmatrix} x_{S*} \\ y_{S*} \\ z_{S*} \end{pmatrix} = \vec{S}^*.$$

Rotating the sun's vector $\vec{S}^*$ according to (6) through the angle i about the x' axis results in

(7) $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{pmatrix} \begin{pmatrix} x_{S*} \\ y_{S*} \\ z_{S*} \end{pmatrix} = \begin{pmatrix} x_{S*} \\ \cos i \; y_{S*} - \sin i \; z_{S*} \\ \sin i \; y_{S*} + \cos i \; z_{S*} \end{pmatrix} =$$
$$- S \begin{pmatrix} \cos d \cos h \sin a - \sin d \cos h \cos a \\ \cos i \sin d \cos h \sin a + \cos i \cos d \cos h \cos a + \sin i \sin h \\ \sin i \sin d \cos h \sin a + \sin i \cos d \cos h \cos a - \cos i \sin h \end{pmatrix} = \begin{pmatrix} x_{S**} \\ y_{S**} \\ z_{S**} \end{pmatrix} = \vec{S}^{**}.$$

The rotation matrices have been chosen in such a way that for the orientation of the dial face the following definitions are applicable:

	positive value	negative value
declination *d*	looking from the zenith: turning the dial face clockwise, away from the east-west direction; the face's normal vector points westwards	looking from the zenith: turning the dial face counterclockwise, away from the east-west direction; the face's normal vector points eastwards
inclination *i*	looking from the east: turning the dial plate clockwise, away from the vertical position	looking from the east: turning the dial plate counterclockwise, away from the vertical position

Shadow casting of the pin gnomon on the dial face

The vector diagram of Fig. 1 delivers $\vec{G} + \vec{S}^{**} = \vec{E}$:

the vector of the pin gnomon with its length G reads $\vec{G} = \begin{pmatrix} x_G \\ y_G \\ z_G \end{pmatrix} = \begin{pmatrix} 0 \\ -G \\ 0 \end{pmatrix}$, the sun's vector $\vec{S}^{**}$ relates to (7) and the vector of the shadow on the face plane has the notation $\vec{E} = \begin{pmatrix} x_E \\ 0 \\ z_E \end{pmatrix}$.

Using these assignments we get the following equations for the parameter S as well as the coordinates x_E and z_E

(8) $-S(\cos d \cos h \sin a - \sin d \cos h \cos a) = x_E$,

(9) $G + S(\cos i \sin d \cos h \sin a + \cos i \cos d \cos h \cos a + \sin i \sin h) = 0$,

(10) $-S(\sin i \sin d \cos h \sin a + \sin i \cos d \cos h \cos a - \cos i \sin h) = z_E$.

Equation (9) reordered with respect to S

(11) $$S = -\frac{G}{\cos i \sin d \cos h \sin a + \cos i \cos d \cos h \cos a + \sin i \sin h},$$

and inserting it in (8) and (10) yields, after an appropriate shaping to the coordinates in demand of the gnomon's shadow tip, to

(12) $$x_E = G\left(\frac{\cos i}{\tan(a-d)} + \frac{\sin i \tan h}{\sin(a-d)}\right)^{-1},$$ (13) $$z_E = G\left(\tan i - \frac{\tan h}{\cos(a-d)}\right) \Big/ \left(1 + \frac{\tan i \tan h}{\cos(a-d)}\right).$$

The pairs of coordinates $x_E(\tau)$, $z_E(\tau)$ using $\delta = const.$ and $x_E(\delta)$, $z_E(\delta)$ using $\tau = const.$, respectively, deliver the graphical representations of the declination lines and hour lines, respectively.

The shadow angle s_G (= hour line angle = angle that the shadow makes with the z axis or its parallel, see Fig. 1) becomes with (8) and (10)

(14) $$s_G = \arctan\left(\frac{x_E}{z_E}\right) = \arctan\left(\frac{\sin(a-d)}{\cos i\,(\cos(a-d)\tan i - \tan h)}\right).$$

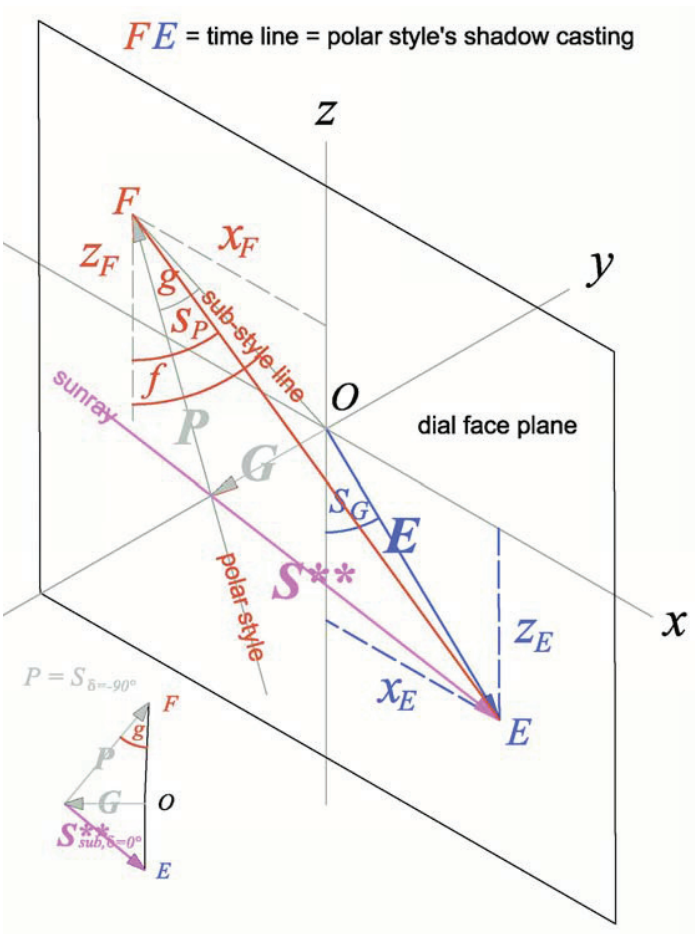

Fig. 1: Geometrical representation of the rotated sun's vector $\vec{S}^{**}$, gnomon vector $\vec{G}$, resultant vector $\vec{E} = \vec{G} + S^{**}$ and polar style vector $\vec{P}$; the further symbols used mean: O = origin of the right-handed coordinate system = pin gnomon's foot point, $E(x_E, z_E)$ = intersection point of the sun's vector onto the dial face plane, $F(x_F, z_F)$ = foot point coordinates of the polar style, s_G = shadow angle of the pin gnomon, s_P = shadow angle of the polar style = time line angle, f = sub-style angle, g = style height.

Characteristic quantities of the polar pointing gnomon

Giving answers to the following questions, it is possible to find out the formulae for the characteristic quantities substyle hour angle τ_{sub} (= hour angle at which the shadow casting is orthogonal onto the dial face), style distance f (= angle that the z axis makes with the orthogonal projection of the polar pointing style onto the dial face), style height g (= angle that the polar pointing style makes with its orthogonal projection onto the dial face), foot point coordinates of the polar pointing style $F(x_F, z_F)$ and time line angle s_P (= angle that the z axis makes with the shadow line of the polar pointing style):

- At which hour angle the absolute value S of the sun's vector is shortest?
- At which sun's declination δ the sun's vector $\vec{S}^{**}$ would be aligned with the polar style vector $\vec{P}$ (see Fig. 1)?
- Which value of the sun's declination δ can be assigned to the foot point coordinates of the polar pointing style?

Substyle hour angle
The differentiation of relation (11) with respect to τ and setting the relevant term equal to zero (= solution for the extreme-value problem) yields

(15) $$\cos i \sin d \cos\delta \cos\tau - (\cos i \cos d \sin\phi \cos\delta + \sin i \cos\phi \cos\delta)\sin\tau = 0$$

and hence finally the substyle hour angle

(16) $$\tau_{sub} = \arctan\left(\frac{\sin d}{\cos d \sin\phi + \tan i \cos\phi}\right)$$ as answer to the first question.

Style distance
The sun's vector cuts perpendicular to the polar style at the equinoxes (δ = $0°$) and is shortest at the substyle hour angle τ_{sub}. Using the following assignments

relation (No.)	cos *h* sin *a* (1)	cos *h* cos *a* (2)	sin *h* (4)
$\delta = 0° \Rightarrow$	$\sin \tau$	$\sin \phi \cos \tau$	$\cos \phi \cos \tau$

the sun's vector (7) reads now

$$(17) \quad \vec{S}^{**}_{sub,\delta=0°} = S^{**}_{sub,\delta=0°} \begin{pmatrix} \cos d \sin \tau_{sub} - \sin d \sin \phi \cos \tau_{sub} \\ \cos i \sin d \sin \tau_{sub} + \cos i \cos d \sin \phi \cos \tau_{sub} + \sin i \cos \phi \cos \tau_{sub} \\ \sin i \sin d \sin \tau_{sub} + \sin i \cos d \sin \phi \cos \tau_{sub} - \cos i \cos \phi \cos \tau_{sub} \end{pmatrix}.$$

It can be shown that the scalar vector product of the two vectors $\vec{S}^{**}$, according to (7), and $\vec{S}^{**}_{sub,\delta=0°}$, according to (17), equals zero if $\delta = \pm 90°$; this mathematical result is evidently because the ends of the polar style each point towards the celestial poles.

Taking into account the following assigments

relation (No.)	cos *h* sin *a* (1)	cos *h* cos *a* (2)	sin *h* (4)
$\delta = -90° \Rightarrow$	*0*	$\cos \phi$	$-\sin \phi$
$\delta = 90° \Rightarrow$	*0*	$-\cos \phi$	$\sin \phi$

the style distance f becomes, with (8) and (10)

$$(18) \quad f = \arctan\left(\frac{x_E(\delta = -90°)}{z_E(\delta = -90°)}\right) = \arctan\left(\frac{-\sin d \cos \phi}{\sin i \cos d \cos \phi + \cos i \sin \phi}\right).$$

Style height

The absolute value of the polar style vector $\vec{P}$ amounts to $P = | S(\delta = -90°)|$

$$(19) \quad S(\delta = -90°) = -\frac{G}{\cos i \cos d \cos \phi - \sin i \sin \phi};$$

therefore the style height g becomes with $\sin g = G / P$, (see Fig. 1)

$$(20) \quad g = \arcsin(\cos i \cos d \cos \phi - \sin i \sin \phi).$$

Foot point coordinates

The foot point coordinates x_F and z_F derive from relations (8) and (10) setting $\delta = -90°$ as well as relation (19):

$$(21) \quad x_F = -G \frac{\sin d \cos \phi}{\cos i \cos d \cos \phi - \sin i \sin \phi}, \qquad (22) \quad z_F = G \frac{\sin i \cos d \cos \phi + \cos i \sin \phi}{\cos i \cos d \cos \phi - \sin i \sin \phi}.$$

Time line angle

The time line angle s_P (see Fig. 1) can be derived by the subsequent execution of the following tasks: at first the differentiation of relations (8) and (10) in consideration of (11) with respect to δ yields $\frac{\mathrm{d}x_E}{\mathrm{d}\delta}$ and $\frac{\mathrm{d}z_E}{\mathrm{d}\delta}$. Next the sun's declination δ has to be set in the functions $\sin \delta$ and $\cos \delta$ equal to $-90°$ so that the above differential quotients change to $\left.\frac{\mathrm{d}x_E}{\mathrm{d}\delta}\right|_{\delta=-90°}$ and $\left.\frac{\mathrm{d}z_E}{\mathrm{d}\delta}\right|_{\delta=-90°}$. Dividing these two terms delivers finally

$$(23) \quad s_P = \arctan\left(\left.\frac{\mathrm{d}x_E}{\mathrm{d}z_E}\right|_{\delta=-90°}\right) = \arctan\left(-\frac{(\cos i \cos \phi - \cos d \sin i \sin \phi)\tan \tau + \sin d \sin i}{\sin d \sin \phi \tan \tau + \cos d}\right).$$

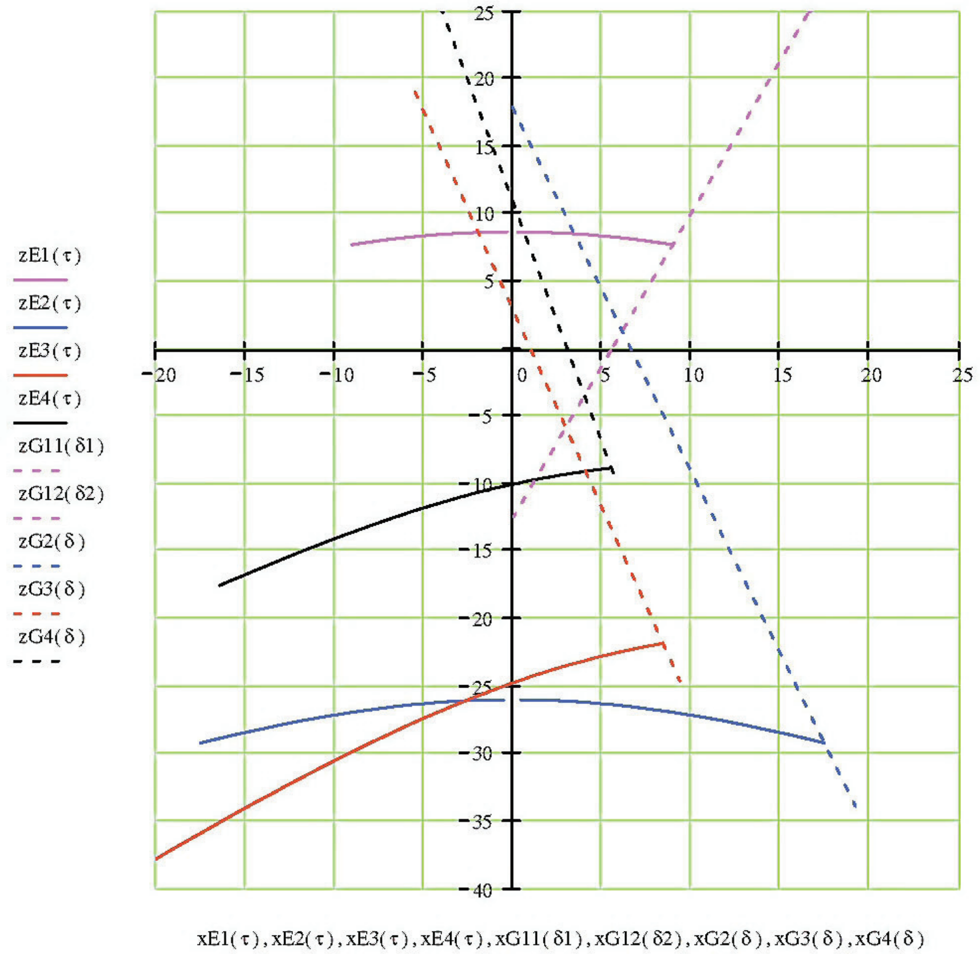

Fig. 2: Graphical representation of the relations (12) and (13); the suffixes mean: *E* = declination lines, *G* = hour lines, *1* = horizontal dial, *2* = vertical south dial, *3* = westward pointed south dial, *4* = declining and inclining dial.

Graphical representations

The common parameters of the curves in Fig. 2 are: $\phi = 50°$, $\delta = 20°$, $\tau = 30°$, $G = 15$. The different parameters are listed in the following table:

type	suffix	*d*	*i*	x_F	z_F
horizontal dial	*11, 12*	*0°*	*90°*	*0*	*-12.586*
vertical south dial	*2*	*0°*	*0°*	*0*	*17.876*
westward pointing south dial	*3*	*20°*	*0°*	*-5,460*	*19.024*
declining and inclining dial	*4*	*20°*	*25°*	*-14.743*	*63.675*

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